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Electromagnetic wave tails in the second approximation to the Einstein–Maxwell equations†

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Abstract. With a finite oscillating linear coherent distribution of electric charge chosen as the source of electromagnetic radiation, it is shown that the outgoing multipole electromagnetic waves of the linear approximation to the Einstein–Maxwell equations produce wave tails in the second approximation which, after the end of the source vibration, constitute *incoming* multipole waves.

1. Introduction

Quite recently considerable attention was given to the properties of wave tails in general relativity. In the case of gravitational waves from an isolated cohesive source, wave tails were found by Bonnor and Rotenberg to occur in the second approximation to the Einstein gravitational field equations§

$$R_{ik} = 0 \tag{1.1}$$

in the form of closed integral expressions (Rotenberg 1964 § 4.7, Bonnor and Rotenberg 1966). Subsequently, Couch *et al.* (1968), Hunter and Rotenberg (1969) established that these wave tails represented an *incoming* multipole wave field imploding at the source. Arising from the mass–multipole interaction, the tails were regarded as the result of back-scattering, or partial reflection, of the outgoing multipole waves (of the linear approximation to (1.1)) by the curvature of the Schwarzschild space caused by the mass of the source (Couch *et al.* 1968).

A purely gravitational system is not the only type of source that produces a curvature of the surrounding space; so does an electromagnetic system. Thus it would be expected that electromagnetic multipole waves from a bounded cohesive source would, likewise, be partially reflected by the curvature of the external Nordström space of the source. It is the purpose of the present paper to confirm this by showing that there do exist electromagnetic wave tails in the second-order monopole–multipole approximation to the empty-space Einstein–Maxwell equations (Eddington 1924 §§ 73 and 77)

$$\begin{aligned} R_{ik} &= -8\pi E_{ik} \qquad & E_k{}^i &= -F^{ia}F_{ka} + \tfrac{1}{4}\delta_k{}^i F^{ab}F_{ab} \\ F^{ia}{}_{;a} &= 0 \qquad & F_{ik} &= \phi_{i,k} - \phi_{k,i} \end{aligned} \tag{1.2}$$

(ϕ_i, F_{ik} and E_{ik} being the electromagnetic 4-potential, 4×4-field and energy tensors, respectively), and that the tails do indeed constitute incoming multipole electromagnetic radiation. To avoid excessive calculation, only the monopole and dipole

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§ Unless otherwise stated or inferred, a Latin index runs from 1 to 4 and a Greek index from 1 to 3; the summation convention applies to both indices. A comma subscript denotes partial differentiation and a semicolon subscript indicates covariant differentiation.

contributions to the multipole wave solution for ϕ_i of the linear approximation to (1.2) will be taken into account, and the source of the electromagnetic wave field will be chosen as a vibrating linear cohesive distribution of charge of finite length carrying negligible mass.

This electric source is described in more detail in § 2, and in § 3 the corresponding (exterior) multipole wave solution for ϕ_i of the linearized Einstein–Maxwell equations is derived in Galilean coordinates (x, y, z, t). In § 4, the method of approximation is presented, and the Bondi metric is introduced in § 5, where also the multipole wave solution for ϕ_i corresponding to this metric is obtained from the one of § 3 in Galilean coordinates. The second approximations to ϕ_i and F_{ik} are calculated in § 6, and the main result, stated above in connection with wave tails, is deduced therein. This is followed by four appendices, which include most of the lengthy calculations.

2. The electromagnetic source

We shall choose as the source a linear coherent charge distribution of finite extent, having mass density small compared with the charge density, along the axis Oz of a (pseudo-Galilean) coordinate system $Oxyz$, the origin O being the centre of mass of the distribution†. The source vibrates arbitrarily but smoothly during a finite interval $t_1 \leqslant t \leqslant t_2$. Thus $\overset{s}{I}(t)$, the sth moment of charge at time t of the distribution about O, is assumed to be a single-valued arbitrary bounded function with unique derivatives of all orders in the interval $t_1 \leqslant t \leqslant t_2$ and constant outside this interval.

3. The solution of the linearized Einstein–Maxwell equations

For ϕ_i representing any electromagnetic field, we obtain here the exterior solution of the linearized form of the second pair of equations (1.2) or of

$$F^{ia}{}_{;a} = 4\pi J^i \qquad F_{ik} = \phi_{i,k} - \phi_{k,i} \tag{3.1}$$

where J_i is the 4-current density of the sources of the field. We then apply the result to the special system of § 2.

Equations (3.1) may be written in the form

$$g^{ab}\phi_{i;ab} + R_i{}^a\phi_a = 4\pi J_i \tag{3.2}$$

connecting the 4-potential ϕ_i with the 4-current density J_i, provided the gauge condition

$$\phi^a{}_{;a} = 0 \tag{3.3}$$

is imposed on ϕ_i (Eddington 1924 § 74). For weak fields, ϕ_i and $R_k{}^i$ are small; thus, in Galilean coordinates $x_i = (x, y, z, t) = (x_\alpha, t)$, the linear approximation to (3.2) and (3.3) gives the linearized wave and gauge equations

$$\eta^{ab}\phi_{i,ab} = 4\pi J_i \qquad \eta^{ab}\phi_{a,b} = 0 \tag{3.4}$$

where

$$\eta^{ik} = \eta_{ik} \overset{\text{def}}{=} \operatorname{diag}(-1, -1, -1, +1). \tag{3.5}$$

The multipole wave solution of these equations (3.4), applicable to the general case

† In the linear approximation to equations (1.2) we shall suppose that distance, time and mass retain their Newtonian meanings.

in which the sources of the field both emit and absorb radiation, is (Rotenberg 1964 § 7.2, 1967)

$$\phi_i = r^{-1}\bar{I}_i + x_\lambda(r^{-2}\hat{I}'_{i/\lambda} + r^{-3}\bar{I}_{i/\lambda}) + \text{terms involving moments of } J_i \text{ of order higher than the first.} \tag{3.6}$$

In this, $r \overset{\text{def}}{=} (x_\alpha x_\alpha)^{1/2}$ is the radial coordinate of the field point P having spherical polar coordinates (r, θ, ϕ); with V being any space volume sufficient to contain the sources of the field, the quantities

$$I_{i/\lambda\mu\nu\ldots}(t) \overset{\text{def}}{=} \int_V x_\lambda x_\mu x_\nu \ldots J_i(x_\alpha, t)\,\mathrm{d}v \qquad (\mathrm{d}v = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3) \tag{3.7}$$

are the moments at time t of the 4-current density J_i for the sources about the coordinate planes $x_\alpha = 0$, and they must satisfy the conservation law

$$\eta^{ab} J_{a,b} = 0 \tag{3.8}$$

for J_i; and a prime indicates differentiation with respect to t. In addition, the notation

$$\bar{\psi} = \alpha\psi(t-r) + \beta\psi(t+r) \qquad \hat{\psi} = \alpha\psi(t-r) - \beta\psi(t+r) \tag{3.9}$$

has been applied to $I_{i/\lambda\mu\nu\ldots}$: the constants α, β in (3.9) are such that $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha+\beta = 1$, and $\alpha : \beta$ represents the comparative strengths between the outgoing and incoming radiation. The solution (3.6) is valid for the external region of the smallest sphere, centre O, that encloses all the sources at all times.

Let e be the total charge of the distribution and a be a constant with the dimensions of length. Introduce the *specific* moments

$$\begin{aligned} h_{\alpha/\lambda_1\lambda_2\ldots\lambda_s} &\overset{\text{def}}{=} e^{-1}a^{-s-1}I_{\alpha/\lambda_1\lambda_2\ldots\lambda_s} \\ h_{4/\lambda_1\lambda_2\ldots\lambda_s} &\overset{\text{def}}{=} e^{-1}a^{-s}I_{4/\lambda_1\lambda_2\ldots\lambda_s} \end{aligned} \tag{3.10}$$

which are unaffected by any change of units in e or a. Then the solution (3.6) yields

$$\begin{aligned} \phi_\alpha &= e\{ar^{-1}\bar{h}_\alpha + \mathrm{O}(a^2)\} \\ \phi_4 &= e\{r^{-1}\bar{h}_4 + ax_\lambda(r^{-2}\hat{h}'_{4/\lambda} + r^{-3}\bar{h}_{4/\lambda}) + \mathrm{O}(a^2)\} \end{aligned} \tag{3.11}$$

with the notation (3.9) applied to $h_{i/\lambda\mu\nu\ldots}$ and a prime denoting differentiation with respect to t.

For presenting the method of approximation in the next section it is convenient to employ a new potential function and a new electromagnetic field function defined by

$$\kappa_i \overset{\text{def}}{=} e\phi_i \qquad H_{ik} \overset{\text{def}}{=} eF_{ik} = \kappa_{i,k} - \kappa_{k,i} \tag{3.12}$$

respectively. Thus (3.11) becomes

$$\begin{aligned} \kappa_\alpha &= \epsilon\{ar^{-1}\bar{h}_\alpha + \mathrm{O}(a^2)\} \\ \kappa_4 &= \epsilon\{r^{-1}\bar{h}_4 + ax_\lambda(r^{-2}\hat{h}'_{4/\lambda} + r^{-3}\bar{h}_{4/\lambda}) + \mathrm{O}(a^2)\} \end{aligned} \tag{3.13}$$

where

$$\epsilon \overset{\text{def}}{=} e^2. \tag{3.14}$$

In (3.13), the 2^s-*pole wave* ($s = 0, 1, 2, \ldots$) is the coefficient of ϵa^s; only the monopole and dipole wave contributions of this multipole wave solution are explicitly shown.

The conservation equation (3.8) may be written as

$$J_{4,4} = J_{\beta,\beta}. \tag{3.15}$$

Multiplying this by x_α and integrating over a space volume V enclosing all the sources of the field, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V x_\alpha J_4 \,\mathrm{d}v = \int_V x_\alpha J_{\beta,\beta}\,\mathrm{d}v = \int_V (x_\alpha J_\beta)_{,\beta}\,\mathrm{d}v - \int_V J_\alpha\,\mathrm{d}v. \tag{3.16}$$

Since $J_i = 0$ on the boundary S of V, it follows from Gauss' theorem that the first integral on the extreme right of (3.16) vanishes, and so, on account of (3.7) and (3.10), $h_\alpha = -h'_{4/\alpha}$. Moreover, it is evident from the second part of (3.10) and from (3.7) that $h_4 = 1$. Hence (3.13) gives

$$\begin{aligned}\kappa_\alpha &= \epsilon\{-ar^{-1}\bar{h}'_{4/\alpha} + \mathrm{O}(a^2)\}\\ \kappa_4 &= \epsilon\{r^{-1} + ax_\lambda(r^{-2}\bar{h}'_{4/\lambda} + r^{-3}\bar{h}_{4/\lambda}) + \mathrm{O}(a^2)\}.\end{aligned} \tag{3.17}$$

The solution (3.13) or (3.17) applies to the region outside the smallest sphere, with centre O, that includes all the sources of the field at all times. As can be verified, this multipole wave solution is, therefore, an exterior solution of the linear approximation

$$\eta^{ab}\kappa_{i,ab} = 0 \qquad \eta^{ab}\kappa_{a,b} = 0 \tag{3.18}$$

to (3.2) ($J_i = 0$) and (3.3), i.e. to (3.1) ($J_i = 0$), i.e. to the second pair of the Einstein–Maxwell equations (1.2).

The equivalence of equations (3.1) to equations (3.2) and (3.3) implies that, in the notation (3.12) and (3.14), the Einstein–Maxwell equations (1.2) may be cast in the form

$$R_{ik} = -8\pi E_{ik} \qquad E_k^{\ i} = \epsilon^{-1}(-H^{ia}H_{ka} + \tfrac{1}{4}\delta_k^{\ i}H^{ab}H_{ab}) \tag{3.19}$$

$$X_i \overset{\text{def}}{=} g^{ab}\kappa_{i;ab} - 8\pi E_i^{\ a}\kappa_a = 0 \qquad X \overset{\text{def}}{=} \kappa^a_{\ ;a} = 0. \tag{3.20}$$

(3.19) and (3.20) will henceforth be referred to as the E and M equations, respectively.

Corresponding to the special system of § 2 the functions $h_{4/\alpha}(t)$, introduced from (3.10) and (3.7), are given by

$$h_{4/1} = h_{4/2} = 0 \qquad h_{4/3} = h(t) \tag{3.21}$$

where eah is the first moment $\overset{1}{I}$ of charge of the source about O. Thus, for *outgoing* waves ($\alpha = 1$, $\beta = 0$) emitted from this special source, the solution (3.17) yields

$$\begin{aligned}\kappa_1 &= \kappa_2 = 0 \qquad \kappa_3 = \epsilon\{-ar^{-1}h' + \mathrm{O}(a^2)\}\\ \kappa_4 &= \epsilon\{r^{-1} + a\cos\theta(r^{-1}h' + r^{-2}h) + \mathrm{O}(a^2)\}\end{aligned} \tag{3.22}$$

in which r, θ are the first two of the spherical polar coordinates of the field point P, $h \equiv h(t-r)$ and a prime denotes differentiation with respect to $t-r$†.

4. The method of approximation

The multipole wave solution (3.17), for the potential κ_i, of the linear approximation (3.18) to the M equations (3.20), describes the *linearized* external electromagnetic field of any bounded cohesive electromagnetic source. The approximate solution is *linear* in ϵ and is also a series in ascending powers of a. These facts suggest that the corresponding exact solution of the M equations (3.20) for κ_i can be expanded as a convergent double series in ascending powers of the two parameters ϵ and a. This, in turn, suggests that an analogous expansion can be made of the solution of the E equations (3.19) for the metric tensor g_{ik} corresponding to κ_i and representing the external gravitational field of the source. Accordingly we write

$$\kappa_i = \sum_{p=1}^{\infty} \sum_{s=0}^{\infty} \epsilon^p a^s \overset{(ps)}{\kappa_i} \tag{4.1}$$

$$g_{ik} = \overset{0}{g}_{ik} + \sum_{p=1}^{\infty} \sum_{s=0}^{\infty} \epsilon^p a^s \overset{(ps)}{g}_{ik} \qquad \left(\overset{0}{g}_{ik} = \sum_{s=0}^{\infty} a^s \overset{(0s)}{g}_{ik}\right). \tag{4.2}$$

In these, $\overset{(ps)}{\kappa_i}$ and $\overset{(ps)}{g}_{ik}$ are independent of ϵ and a; and $\overset{0}{g}_{ik}$, independent of ϵ, refers to the external gravitational field due only to the mass distribution of the source. As in § 2, let us consider the case in which the mass density of the source is small compared with its charge density; then we may ignore the former and the term $\overset{0}{g}_{ik}$ on the right of equation (4.2) reduces to $\overset{(00)}{g}_{ik}$, the part of g_{ik} representing flat space-time. Thus equation (4.2) now becomes

$$g_{ik} = \overset{(00)}{g}_{ik} + \sum_{p=1}^{\infty} \sum_{s=0}^{\infty} \epsilon^p a^s \overset{(ps)}{g}_{ik}. \tag{4.3}$$

Expansions similar to equations (4.1) and (4.3) clearly apply to κ^i and g^{ik}.

By virtue of the second members of equations (3.12) and (3.19) it is evident that the expansions (4.1) and (4.3) produce analogous expansions for H_{ik} and E_{ik} (together with their associate contravariant and mixed tensors). Thus

$$H_{ik} = \sum_{p=1}^{\infty} \sum_{s=0}^{\infty} \epsilon^p a^s \overset{(ps)}{H}_{ik} \qquad E_{ik} = \sum_{p=1}^{\infty} \sum_{s=0}^{\infty} \epsilon^p a^s \overset{(ps)}{E}_{ik} \qquad \ldots \tag{4.4}$$

$\overset{(ps)}{H}_{ik}$, $\overset{(ps)}{E}_{ik}$, ... being independent of ϵ and a.

Let us insert equations (4.1), (4.3) and the second of equations (4.4) in equations (3.19) and (3.20), extract the coefficients of $\epsilon^p a^s$ from equations (3.19) and similarly from equations (3.20), and equate the two sets of coefficients to zero. Then we obtain ten second-order differential equations of the form

$$\Phi_{lm}(\overset{(ps)}{g}_{ik}) = \overset{(ps)}{\Psi}_{lm}(\overset{(qr)}{g}_{ik}) + \text{constant} \times \overset{(ps)}{E}_{lm} \qquad (1 \leqslant q \leqslant p-1,\ 0 \leqslant r \leqslant s) \tag{4.5}$$

hereafter referred to as the (ps) (*approximation to the*) *E equations*, and five differential

† From (3.6), (3.7) and the first part of (3.12), κ_i ($i = 1, 2$) vanish completely in (3.22) because $J_i = 0$ ($i = 1, 2$) for this particular source.

equations of the form

$$\Phi_l(\overset{(ps)}{\kappa_i}) = \overset{(ps)}{\Psi_l}(\overset{(qr)}{\kappa_i}, \overset{(cd)}{g_{ik}}) + \text{constant} \times \sum_{m=1}^{p-1} \sum_{n=0}^{s} \overset{(mn)}{E_l^a} \overset{(p-ms-n)}{\kappa_a}$$

$$\Phi(\overset{(ps)}{\kappa_i}) = \overset{(ps)}{\Psi}(\overset{(qr)}{\kappa_i}, \overset{(cd)}{g_{ik}}) \tag{4.6}$$

$$(1 \leqslant q \leqslant p-1, \quad 0 \leqslant r \leqslant s; \quad 1 \leqslant c \leqslant p-1, \quad 0 \leqslant d \leqslant s)$$

hereafter referred to as the (ps) (*approximation to the*) *M equations,* the first four being of the second order and the fifth being of the first order. In equations (4.5), Φ_{lm} on the left are linear in $\overset{(ps)}{g_{ik}}$ (and their derivatives); $\overset{(ps)}{\Psi_{lm}}$ on the right are nonlinear in $\overset{(qr)}{g_{ik}}$ (and their derivatives), known from earlier approximations to equations (3.19). In equations (4.6), Φ_l and Φ on the left are linear in $\overset{(ps)}{\kappa_i}$ (and their derivatives); $\overset{(ps)}{\Psi_l}$ and $\overset{(ps)}{\Psi}$ on the right consist of nonlinear terms each having as factors one of $\overset{(qr)}{\kappa_i}$ (and their derivatives), known from earlier approximations to equations (3.20), and at least one of $\overset{(cd)}{g_{ik}}$ (and their derivatives), known from earlier approximations to equations (3.19). Solving the (ps) field equations (4.5) and (4.6) thus determines $\overset{(ps)}{g_{ik}}$ and $\overset{(ps)}{\kappa_i}$ in terms of previously calculated $\overset{(qr)}{g_{ik}}$ and $\overset{(qr)}{\kappa_i}$.

For $p = 1$, none of the nonlinear expressions $\overset{(ps)}{\Psi_{lm}}$, $\overset{(ps)}{\Psi_l}$, $\overset{(ps)}{\Psi}$ and the double summation expression on the right of equations (4.5) and (4.6) has any terms. Consequently, for any $s \geqslant 0$ the set of $(1s)$ M equations is linear and homogeneous in $\overset{(1s)}{\kappa_i}$ and their derivatives. The collection of all the sets of $(1s)$ M equations ($s = 0, 1, 2, \ldots$) constitutes the linear approximation to the M equations (3.20), namely equations (3.18) in Galilean coordinates; their solutions for $\overset{(1s)}{\kappa_i}$, in these coordinates, are the 2^s-pole wave solutions given by the coefficients of ϵa^s in the full version of the solution (3.13). Accordingly, for any given $p \geqslant 1$, the sets of (ps) M equations ($s = 0$, $1, 2, \ldots$) will be considered as making up the pth approximation to the M equations (3.20). Similarly, for any given $p \geqslant 1$, the sets of (ps) E equations ($s = 0, 1$, $2, \ldots$) will be regarded as constituting the pth approximation to the E equations (3.19).

For $p \geqslant 2$, the solution of the (ps) M equations (4.6) is indeterminate to the extent of a complementary solution of these equations (4.6), that is, a solution of

$$\Phi_l(\overset{(ps)}{\kappa_i}) = 0 \qquad \Phi(\overset{(ps)}{\kappa_i}) = 0. \tag{4.7}$$

Nevertheless, we shall assume that the functions representing the essential sources of the κ_i wave field have already been chosen for the 2^s-pole wave solutions of the linear, $(1s)$, M equations

$$\Phi_l(\overset{(1s)}{\kappa_i}) = 0 \qquad \Phi(\overset{(1s)}{\kappa_i}) = 0. \tag{4.8}$$

No new source functions are to be employed other than the ones required to ensure the satisfaction of the inhomogeneous equations (4.6) together with the regularity conditions outlined in the last paragraph of Appendix 3. In § 6, this rule and the corresponding one for the solution $\overset{(ps)}{g_{ik}}$ of the (ps) E equations will accordingly be adopted; a similar convention was adhered to in previous works for purely gravitational sources (Bonnor 1959, Rotenberg 1964 § 4.3, 1968, Bonnor and Rotenberg 1966).

The progress of successive approximations to g_{ik} as solutions of the consecutive sets of (ps) E equations depends on the progress of successive approximations to κ_i as solutions of the consecutive sets of (ps) M equations, and vice versa. This can be explained in the following way. Next after the $\overset{(00)}{g}_{ik}$ contribution (for flat space-time) on the right of equation (4.3) come the $\overset{(1s)}{g}_{ik}$ contributions. These can be calculated from equation (4.5) ($p = 1$) only after the $\overset{(1s)}{\kappa}_i$ contributions have been obtained (by the method of § 3 or directly from equations (4.6) ($p = 1$)); this is evident from the presence of the $\overset{(1s)}{E}_{lm}$ term on the right of equation (4.5) ($p = 1$), which involves $\overset{(1s)}{\kappa}_i$. The $\overset{(2s)}{\kappa}_i$, coming next after $\overset{(1s)}{\kappa}_i$ on the right of equation (4.1), cannot be derived from equations (4.6) ($p = 2$) until $\overset{(1s)}{g}_{ik}$, entering $\overset{(2s)}{\Psi}_l$ and $\overset{(2s)}{\Psi}$, are known. The $\overset{(2s)}{g}_{ik}$, immediately following $\overset{(1s)}{g}_{ik}$ on the right of equation (4.3), cannot be determined from equation (4.5) ($p = 2$) until $\overset{(2s)}{\kappa}_i$ as well as $\overset{(1s)}{\kappa}_i$, absorbed in $\overset{(2s)}{E}_{lm}$ on the right of equation (4.5) ($p = 2$), are known—and so on indefinitely. In short, $\overset{(1s)}{\kappa}_i$ are needed to know $\overset{(1s)}{g}_{ik}$, which in turn are needed to know $\overset{(2s)}{\kappa}_i$, which in turn are needed to know $\overset{(2s)}{g}_{ik}$, ... Thus the problem of solving the successive sets of (ps) E equations (4.5) and that of solving the successive sets of $(ps)\,M$ equations (4.6) cannot be isolated from each other.

5. The metric

For the special axi-symmetric source of § 2 we shall henceforth employ the following axi-symmetric metric due to Bondi (1960)

$$ds^2 = -r^2(B\,d\theta^2 + C\sin^2\theta\,d\phi^2) + D\,du^2 + 2F\,dr\,du + 2rG\,d\theta\,du \qquad C = B^{-1}. \quad (5.1)$$

In this, (r, θ, ϕ) are spherical polar coordinates of the field point $\mathbf{P}$; the time-like coordinate u = retarded time $t-r$; and B, C, D, F, G are dependent on r, θ, u only. In accordance with the expansion (4.3), the corresponding non-zero g_{ik} will have the expansions

$$\begin{aligned}
-g_{22} &= r^2B &&= r^2\Big(1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty}\epsilon^p a^s \overset{(ps)}{B}\Big)\\
-g_{33} &= r^2\sin^2\theta\,C &&= r^2\sin^2\theta\Big(1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty}\epsilon^p a^s \overset{(ps)}{C}\Big)\\
g_{44} &= D &&= 1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty}\epsilon^p a^s \overset{(ps)}{D}\\
g_{14} &= F &&= 1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty}\epsilon^p a^s \overset{(ps)}{F}\\
g_{24} &= rG &&= r\sum_{p=1}^{\infty}\sum_{s=0}^{\infty}\epsilon^p a^s \overset{(ps)}{G}
\end{aligned} \qquad (5.2)$$

in which $\overset{(ps)}{B}, \ldots, \overset{(ps)}{G}$ are functions of (r, θ, u), and $\overset{(ps)}{C}$ (p, s given) is connected with $\overset{(qr)}{B}$ ($1 \leqslant q \leqslant p$, $0 \leqslant r \leqslant s$) by the second of equations (5.1). The leading terms on the extreme right of equations (5.2) constitute the nonvanishing components of $\overset{(00)}{g}_{ik}$, which refer to flat space–time, namely

$$\overset{(00)}{g}_{22} = -r^2 \qquad \overset{(00)}{g}_{33} = -r^2 \sin^2\theta \qquad \overset{(00)}{g}_{44} = 1 \qquad \overset{(00)}{g}_{14} = 1 \tag{5.3}$$

by virtue of equation (A.5) ($m = 0$) of Appendix 1.

Carrying out the coordinate transformation

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \qquad t = u+r \tag{5.4}$$

on equations (3.22) we obtain

$$\begin{aligned} &\kappa_1 = \epsilon\{r^{-1} + a\cos\theta\, r^{-2}h + \mathrm{O}(a^2)\} \qquad \kappa_2 = \epsilon\{a\sin\theta h' + \mathrm{O}(a^2)\} \\ &\kappa_3 = 0 \qquad \kappa_4 = \epsilon\{r^{-1} + a\cos\theta\,(r^{-1}h' + r^{-2}h) + \mathrm{O}(a^2)\} \end{aligned} \tag{5.5}$$

(where $h \equiv h(u)$, $h' \equiv h'(u)$) as the multipole wave solution of the linear approximation to equations (3.20) in coordinates of the Bondi metric (5.1)†.

We shall assume that, like (5.5), the corresponding exact solution of (3.20) gives κ_i as functions of r, θ, u only and

$$\kappa_3 = 0. \tag{5.6}$$

The remaining κ_i will be expanded in the form (4.1) as

$$\begin{aligned} \kappa_1 &= \alpha = \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} \epsilon^p a^s \overset{(ps)}{\alpha} \\ \kappa_2 &= r\beta = r\sum_{p=1}^{\infty}\sum_{s=0}^{\infty} \epsilon^p a^s \overset{(ps)}{\beta} \\ \kappa_4 &= \delta = \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} \epsilon^p a^s \overset{(ps)}{\delta} \end{aligned} \tag{5.7}$$

where $\alpha, \ldots$ and $\overset{(ps)}{\alpha}, \ldots$ are functions of (r, θ, u).

Finally, in terms of the retarded time u, the period of vibration of the source will be denoted by $u_1 \leqslant u \leqslant u_2$. This corresponds to the original notation $t_1 \leqslant t \leqslant t_2$ in § 2.

6. The (21) approximation to the κ_i and H_{ik} field. Wave tails

Putting $p = 2$, $s = 1$ in equations (4.6) we have the (21) M equations in the form

$$\begin{aligned} \Phi_l(\overset{(21)}{\kappa_i}) &= \overset{(21)}{\Psi_l}(\overset{(10)}{\kappa_i}, \overset{(11)}{\kappa_i}; \overset{(10)}{g_{ik}}, \overset{(11)}{g_{ik}}) + \text{constant} \times (\overset{(10)}{\kappa_a}\, \overset{(11)}{E_l^a} + \overset{(11)}{\kappa_a}\, \overset{(10)}{E_l^a}) \\ \Phi(\overset{(21)}{\kappa_i}) &= \overset{(21)}{\Psi}(\overset{(10)}{\kappa_i}, \overset{(11)}{\kappa_i}; \overset{(10)}{g_{ik}}, \overset{(11)}{g_{ik}}) \end{aligned} \tag{6.1}$$

† An additional sequence of transformations of the form

$$x^i = x^{*i} + \epsilon^p a^s \overset{(ps)}{\xi^i}(x^*) \qquad (x^i = r, \theta, \phi, u; p \geqslant 1, s \geqslant 0)$$

normally required to ensure that a metric in coordinates (r, θ, ϕ, u) satisfies the conditions $g_{11} = g_{12} = 0$, $g_{22}g_{33} = r^4\sin^2\theta$ of the Bondi metric, would have the effect of introducing in the solution (5.5) only nonlinear terms, of order $\epsilon^{p+1}a^s$ ($p \geqslant 1$, $s \geqslant 0$), the κ_i in equations (5.5) being of order ϵ.

in which the expressions $\overset{(21)}{\Psi}_l$, $\overset{(21)}{\Psi}$ on the right are composed of terms coming from the combinations

$$\overset{(10)}{\kappa}_i \times \overset{(11)}{g}_{ik} \qquad \overset{(11)}{\kappa}_i \times \overset{(10)}{g}_{ik} \tag{6.2}$$

of $\overset{(1r)}{\kappa}_i$, $\overset{(1r)}{g}_{ik}$ and their derivatives. Now, $\overset{(1s)}{E}_{ik}$ consist of products of the first derivatives of $\overset{(1r)}{\kappa}_i$ $(0 \leqslant r \leqslant s)$ and, consequently from the right of equations (4.5) $(p = 1)$, $\overset{(10)}{g}_{ik}$ depend on $\overset{(10)}{\kappa}_i$ and $\overset{(11)}{g}_{ik}$ depend on $\overset{(10)}{\kappa}_i$ and $\overset{(11)}{\kappa}_i$. It therefore follows from the combinations (6.2) that the right-hand sides of equations (6.1) are made up of combinations among $\overset{(1s)}{\kappa}_i$ $(s = 0, 1)$ and quantities depending on $\overset{(1s)}{\kappa}_i$ $(s = 0, 1)$. Accordingly, we speak of the (21) M equations (6.1) as consisting entirely of the '(10)–(11)' or 'monopole–dipole' interaction.

We now derive formulae for $\overset{(1s)}{\kappa}_i$, $\overset{(1s)}{g}_{ik}$, $\overset{(1s)}{E}_k{}^i$ $(s = 0, 1)$, which occur on the right of these (21), monopole–dipole, M equations (6.1). The $\overset{(1s)}{\kappa}_i$ $(s = 0, 1)$ can be obtained immediately from equations (5.5); they are

$$\overset{(10)}{\kappa}_1 = r^{-1} \qquad \overset{(10)}{\kappa}_2 = 0 \qquad \overset{(10)}{\kappa}_3 = 0 \qquad \overset{(10)}{\kappa}_4 = r^{-1} \tag{6.3}$$

$$\overset{(11)}{\kappa}_1 = \cos\theta\, r^{-2}h \qquad \overset{(11)}{\kappa}_2 = \sin\theta\, h' \qquad \overset{(11)}{\kappa}_3 = 0 \qquad \overset{(11)}{\kappa}_4 = \cos\theta\,(r^{-1}h' + r^{-2}h). \tag{6.4}$$

Next we use these in the second of equations (3.12) to calculate $\overset{(1s)}{H}_{ik}$ $(s = 0, 1)$, and then $\overset{(1s)}{H}{}^{ik}$ $(s = 0, 1)$ by means of equations (5.3). Inserting the resulting values for $\overset{(1s)}{H}_{ik}$, $\overset{(1s)}{H}{}^{ik}$ $(s = 0, 1)$ in the $(1s)$ approximations $(s = 0, 1)$ to the second of equations (3.19), we find for the nonvanishing $\overset{(1s)}{E}_k{}^i$ $(s = 0, 1)$

$$\overset{(10)}{E}_1{}^1 = -\overset{(10)}{E}_2{}^2 = -\overset{(10)}{E}_3{}^3 = \overset{(10)}{E}_4{}^4 = \tfrac{1}{2}r^{-4} \tag{6.5}$$

$$\overset{(11)}{E}_1{}^1 = -\overset{(11)}{E}_2{}^2 = -\overset{(11)}{E}_3{}^3 = \overset{(11)}{E}_4{}^4 = 2\cos\theta\,(r^{-4}h' + r^{-5}h)$$

$$\overset{(11)}{E}_2{}^1 = \sin\theta\,(r^{-2}h'' + r^{-3}h' + r^{-4}h) \qquad \overset{(11)}{E}_1{}^2 = \sin\theta\, r^{-6}h \tag{6.6}$$

$$\overset{(11)}{E}_4{}^2 = -\sin\theta\,(r^{-4}h'' + r^{-5}h') \qquad \overset{(11)}{E}_2{}^4 = -\sin\theta\, r^{-4}h.$$

Finally, the solutions $\overset{(1s)}{g}_{ik}$ of the $(1s)$ approximations to equations (3.19) may be got on insertion of equations (A.13) in equations (A.14) to (A.17) (Appendix 2) and employment of the appropriate formulae for $\overset{(1s)}{E}_k{}^i$. Corresponding to the values (6.5) and (6.6) for $\overset{(10)}{E}_k{}^i$ and $\overset{(11)}{E}_k{}^i$, the (10) and (11) solutions (Galilean at spatial infinity and satisfying the regularity condition (A.18)) turn out to be the following nonzero $\overset{(1s)}{g}_{ik}$ $(s = 0, 1)$:

$$\overset{(10)}{g}_{44} = 4\pi r^{-2} \tag{6.7}$$

$$\overset{(11)}{g}_{44} = \tfrac{8}{3}\pi\cos\theta\,(4r^{-2}h' + 3r^{-3}h) \qquad \overset{(11)}{g}_{24} = \tfrac{4}{3}\pi\sin\theta\,(4r^{-1}h' - 3r^{-2}h). \tag{6.8}$$

(It will be noticed that the (10) metric (6.7) checks with that obtained from equation (A.5), $m = 0$, with the use of equations (A.2) and (3.14).)

We can now proceed with the tedious but straightforward calculation of the (21), monopole–dipole, approximation to equations (3.20) with the aid of equations (6.3) to (6.8). The result is equations (A.19) to (A.22), of Appendix 3, with $\overset{(21)}{U}$, $\overset{(21)}{V}$, $\overset{(21)}{W}$, $\overset{(21)}{T}$ on the right given by

$$\overset{(21)}{U} = \tfrac{16}{3}\pi\cos\theta\,(16r^{-5}h' + 15r^{-6}h) \qquad \overset{(21)}{V} = \tfrac{16}{3}\pi\sin\theta\,(r^{-5}h' + 3r^{-6}h)$$
$$\overset{(21)}{W} = \tfrac{16}{3}\pi\cos\theta\,(4r^{-4}h'' + r^{-5}h') \qquad \overset{(21)}{T} = \tfrac{16}{3}\pi\cos\theta\,(4r^{-5}h' + 3r^{-6}h). \tag{6.9}$$

A solution of these equations, satisfying the Galilean condition at spatial infinity and the regularity condition (A.25), is found with the help of equations (A.21), (A.23) and (A.24) to be

$$\overset{(21)}{\alpha} = -\tfrac{4}{3}\pi\cos\theta\{9r^{-4}h + 2(r^{-1}K)_1\} \qquad \overset{(21)}{\beta} = \tfrac{4}{3}\pi\sin\theta\,(-3r^{-4}h + r^{-1}K_1)$$
$$\overset{(21)}{\delta} = \tfrac{4}{3}\pi\cos\theta\{4r^{-3}h' - r(r^{-2}K)_1\} \tag{6.10}$$

where

$$K \overset{\text{def}}{=} \frac{\partial}{\partial u}\int_\infty^r w^{-2}h(u+2r-2w)\,dw = -\tfrac{1}{2}r^{-2}h - \int_\infty^r w^{-3}h(u+2r-2w)\,dw. \tag{6.11}$$

The function K does not become static immediately after the end of the vibration of the source ($u = u_2$) and is therefore called a 'wave tail'.

The (21) contributions to the nonvanishing components H_{ik} are, because of equations (5.7), (6.10) and the second of equations (3.12)

$$\overset{(21)}{H}_{12} = -\tfrac{4}{3}\pi\sin\theta\, r(r^{-1}K)_{11} \qquad \overset{(21)}{H}_{14} = -\tfrac{8}{3}\pi\cos\theta\,(r^{-2}K)_1$$
$$\overset{(21)}{H}_{24} = \tfrac{2}{3}\pi\sin\theta\,(4r^{-3}h' + K_{11} - 2r^{-1}K_1 + 4r^{-2}K) \tag{6.12}$$

which also contain the wave-tail function K†.

For $u \geqslant u_2$, equations (6.10) and (6.12) become

$$\overset{(21)}{\alpha} = -\tfrac{4}{3}\pi\cos\theta\{9r^{-4}h + 2(r^{-1}K)_1\} \qquad \overset{(21)}{\beta} = \tfrac{4}{3}\pi\sin\theta\,(-3r^{-4}h + r^{-1}K_1)$$
$$\overset{(21)}{\delta} = -\tfrac{4}{3}\pi\cos\theta\, r(r^{-2}K)_1 \tag{6.13}$$

and

$$\overset{(21)}{H}_{12} = -\tfrac{4}{3}\pi\sin\theta\, r(r^{-1}K)_{11} \qquad \overset{(21)}{H}_{14} = -\tfrac{8}{3}\pi\cos\theta\,(r^{-2}K)_1$$
$$\overset{(21)}{H}_{24} = \tfrac{2}{3}\pi\sin\theta\,(K_{11} - 2r^{-1}K_1 + 4r^{-2}K) \tag{6.14}$$

respectively. As can be verified, the terms involving h explicitly on the right of equations (6.13) form a solution for the static potential due to the static terms of the

† It was the suggestion of a referee to check that the tails found in the potential κ_i imply their existence in the field H_{ik} as well, as only H_{ik} is observable. The referee mentioned a paper by Künzle (1968), in which the possibility was demonstrated that the potential alone, and not the field, possesses tails. In such cases the tails are of no physical consequence.

monopole–dipole interaction, given by equations (A.19) to (A.22) and (6.9) with exclusion of terms containing time derivatives. With regard to the remaining terms on the right of equations (6.13) and all the terms on the right of equation (6.14), involving the tail function K, we note that for $u \geqslant u_2$

$$K_1 - 2K_4 = r^{-2}h' = 0. \tag{6.15}$$

Hence K is a function of $u+2r$ after the end of the vibration of the source. So comparing the contribution containing K on the right of equations (6.13) with the right of equations (A.28) of Appendix 4, and comparing equations (6.14) with equations (A.29), we arrive at the following result:

Electromagnetic wave tails appear in the (21) monopole–dipole approximation to the Einstein–Maxwell equations which, after the end of the source vibration, represent an incoming dipole wave field; this incoming dipole wave corresponds to a source similar to the one in § 2 with first moment $\frac{8}{3}\pi e^3 aK$ about its centre of mass.

Appendix 1. The Nordström–Bondi metric

The Nordström solution in standard form for a static, spherically symmetric distribution of matter and charge, of total mass m and with total charge e, is

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}+\frac{\mu^2}{r^2}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)+\left(1-\frac{2m}{r}+\frac{\mu^2}{r^2}\right)\mathrm{d}t^2 \tag{A.1}$$

where

$$\mu \overset{\text{def}}{=} 2\pi^{1/2}e. \tag{A.2}$$

The coordinate transformation

$$t = (u+r)+f(r) \qquad r > 2m \tag{A.3}$$

with

$$f(r) = \begin{cases} m\ln(r^2-2mr+\mu^2) - \dfrac{\mu^2-2m^2}{(\mu^2-m^2)^{1/2}}\tan^{-1}\dfrac{r-m}{(\mu^2-m^2)^{1/2}} & m<\mu \\[2ex] 2m\ln(r-m)-\dfrac{m^2}{r-m} & m=\mu \\[2ex] m\ln(r^2-2mr+\mu^2)+\dfrac{2m^2-\mu^2}{2(m^2-\mu^2)^{1/2}}\ln\dfrac{r-m-(m^2-\mu^2)^{1/2}}{r-m+(m^2-\mu^2)^{1/2}} & m>\mu \end{cases} \tag{A.4}$$

brings (A.1) into the Nordström–Bondi metric

$$\mathrm{d}s^2 = -r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)+\left(1-\frac{2m}{r}+\frac{\mu^2}{r^2}\right)\mathrm{d}u^2+2\,\mathrm{d}r\,\mathrm{d}u. \tag{A.5}$$

Here, u is essentially the retarded time $t-r$; this is readily seen from (A.3) and the fact that $f(r)$ is small.

Appendix 2. The approximate E equations and their solution

In the notation (5.2), the (ps) approximation to equations (3.19) for the Bondi metric (5.1) yields the seven equations below, the labels (ps) not appearing above the capital letters (as they ought to) to save printing. A subscript 1, 2 or 4 after B, D, F or G means differentiation with respect to r, θ or u, respectively—a notation to apply to any nontensorial symbol, unless the contrary is implied. Finally, $R'_{ik} \overset{\text{def}}{=} R_{ik}+8\pi E_{ik}$.

$$2R'_{11} = 0:\ -4r^{-1}F_1 \qquad = P \tag{A.6}$$

$$\begin{aligned}2r^{-2}R'_{22} = 0:\ & B_{11}-2B_{14}+2r^{-1}(B_1-B_4+D_1-F_1-G_{12})\\ & +r^{-2}(-B_{22}-3B_2\cot\theta+2B+2D\\ & +2F_{22}-4F-4G_2-2G\cot\theta) \qquad = Q\end{aligned} \tag{A.7}$$

$$\begin{aligned}2r^{-2}\operatorname{cosec}^2\theta R'_{33} = 0:\ & -B_{11}+2B_{14}+2r^{-1}(-B_1+B_4+D_1-F_1-G_1\cot\theta)\\ & +r^{-2}(-B_{22}-3B_2\cot\theta+2B+2D\\ & +2F_2\cot\theta-4F-2G_2-4G\cot\theta) \qquad = R\end{aligned} \tag{A.8}$$

$$\begin{aligned}2R'_{44} = 0:\ & -D_{11}+2F_{14}+2r^{-1}(-D_1-D_4+2F_4+G_{24}+G_4\cot\theta)\\ & -r^{-2}(D_{22}+D_2\cot\theta) \qquad = S\end{aligned} \tag{A.9}$$

$$\begin{aligned}2r^{-1}R'_{12} = 0:\ & -G_{11}+r^{-1}(-B_{12}-2B_1\cot\theta+F_{12}-2G_1)\\ & +2r^{-2}(-F_2+G) \qquad = L\end{aligned} \tag{A.10}$$

$$\begin{aligned}2R'_{14} = 0:\ & -D_{11}+2F_{14}+r^{-1}(-2D_1+G_{12}+G_1\cot\theta)\\ & +r^{-2}(-F_{22}-F_2\cot\theta+G_2+G\cot\theta) \qquad = M\end{aligned} \tag{A.11}$$

$$\begin{aligned}2r^{-1}R'_{24} = 0:\ & -G_{11}+G_{14}+r^{-1}(-B_{24}-2B_4\cot\theta-D_{12}\\ & +F_{12}+F_{24}-2G_1-G_4) \qquad = N.\end{aligned} \tag{A.12}$$

The left-hand sides of these equations, linear in $\overset{(ps)}{g}_{ik}$ (and their derivatives), correspond to Φ_{lm} on the left of equations (4.5), and the quantities $P, Q, \ldots, N$ on the right-hand sides correspond to the right of equations (4.5) and are known from lower approximations to equations (3.19). For $p = 1$, Ψ'_{lm} on the right of equations (4.5) vanish and only the terms involving E_{ik} survive there. In fact $P, Q, \ldots, N$ on the right of equations (A.6) to (A.12) are given by

$$\begin{aligned}&\overset{(1s)}{P} = -16\pi\overset{(1s)}{E_1}{}^4 \quad \overset{(1s)}{Q} = 16\pi\overset{(1s)}{E_2}{}^2 \quad \overset{(1s)}{R} = 16\pi\overset{(1s)}{E_3}{}^3 \quad \overset{(1s)}{S} = -16\pi(\overset{(1s)}{E_4}{}^1+\overset{(1s)}{E_4}{}^4)\\ &\overset{(1s)}{L} = -16\pi r^{-1}\overset{(1s)}{E_2}{}^4 \quad \overset{(1s)}{M} = -16\pi\overset{(1s)}{E_4}{}^4 \quad \overset{(1s)}{N} = 16\pi r\overset{(1s)}{E_4}{}^2.\end{aligned} \tag{A.13}$$

The formal solution of equations (A.6) to (A.12) has been derived in works by Bonnor, Hunter and Rotenberg (Rotenberg 1964 Appendix D-1, 1966, Bonnor and Rotenberg 1966, Hunter and Rotenberg 1969). It is given as

$$F = -\tfrac{1}{4}\int rP\,\mathrm{d}r+\eta(\theta, u) \tag{A.14}$$

$$\Box' D \stackrel{\text{def}}{=} D_{11} - 2D_{14} + 2r^{-1}(D_1 + D_4) + r^{-2}(D_{22} + D_2 \cot\theta)$$
$$= -S + 2(F_{14} + 2r^{-1}F_4)$$
$$+ 2r^{-2}[\int \{r^2(M - 2F_{14}) + (F_{22} + F_2 \cot\theta)\}\, dr + \chi(\theta, u)]_4 \qquad \text{(A.15)}$$

$$G = r^{-1}\int F_2\, dr + r^{-1}\operatorname{cosec}\theta \int \sin\theta\{\int r^2(M - 2F_{14})\, dr + r^2 D_1 + \chi\}\, d\theta$$
$$+ \nu(r, u)\operatorname{cosec}\theta \qquad \text{(A.16)}$$

$$B = \operatorname{cosec}^2\theta \int \sin^2\theta[-\int\{rL + 2r^{-1}(F_2 - G)\}\, dr + F_2 - G - rG_1]\, d\theta$$
$$+ \tau(r, u)\operatorname{cosec}^2\theta + \mu(\theta, u) \qquad \text{(A.17)}$$

where $\eta(\theta, u)$, $\chi(\theta, u)$, $\nu(r, u)$, $\tau(r, u)$, $\mu(\theta, u)$ are five functions of integration. These functions must be chosen to render $\overset{(ps)}{g}_{ik}$ Galilean at spatial infinity, and nonsingular along the axis Oz of symmetry except at O. A sufficient condition for the (ps) metric to be regular along Oz ($r > 0$) is that

$$B\operatorname{cosec}^2\theta,\ C\operatorname{cosec}^2\theta,\ D,\ F,\ G\operatorname{cosec}\theta \text{ be of class } C^2 \text{ near } \sin\theta = 0. \qquad \text{(A.18)}$$

The solution must be verified to satisfy all the seven equations (A.6) to (A.12), and it may be found that further conditions are imposed by these field equations on the five functions of integration.

Appendix 3. The approximate *M* equations and their solution

The (ps) approximation to equations (3.20) for the Bondi metric (5.1) consists of the following four equations, which employ the notation (5.7):

$$X_1 = 0:\ -\alpha_{11} + 2\alpha_{14} + 2r^{-1}(-\alpha_1 + \alpha_4)$$
$$+ r^{-2}(-\alpha_{22} - \alpha_2\cot\theta + 2\alpha + 2\beta_2 + 2\beta\cot\theta - 2\delta) \quad = U \qquad \text{(A.19)}$$
$$X_2 = 0:\ -\beta_{11} + 2\beta_{14} + 2r^{-1}(-\beta_1 + \beta_4)$$
$$+ r^{-2}(-2\alpha_2 - \beta_{22} - \beta_2\cot\theta + \beta\operatorname{cosec}^2\theta + 2\delta_2) \quad = V \qquad \text{(A.20)}$$

$$X_4 = 0:\ \Box\delta \stackrel{\text{def}}{=} \delta_{11} - 2\delta_{14} + 2r^{-1}(\delta_1 - \delta_4) + r^{-2}(\delta_{22} + \delta_2\cot\theta) = W \qquad \text{(A.21)}$$
$$X = 0:\ r^{-1}(\alpha_1 - \alpha_4 - \delta_1) + r^{-2}(2\alpha + \beta_2 + \beta\cot\theta - 2\delta) \quad = T. \qquad \text{(A.22)}$$

To save printing the labels (ps) have been omitted, as in Appendix 2. The left-hand sides of equations (A.19) to (A.22), linear in $\overset{(ps)}{\kappa}_i$ (and their derivatives), correspond to Φ_i and Φ on the left of equations (4.6); U, V, W, T on the right of equations (A.19) to (A.22) correspond to the right of equations (4.6), and are determined from earlier approximations to (3.20). When $p = 1$, U, V, W, T vanish and then, as can be verified, a solution of the above equations is the 2^s-pole wave solution, the ϵa^s contribution in the expanded version of (5.5).

Eliminating β between equations (A.19) and (A.22) we obtain

$$\alpha_{11} - 2\alpha_{14} + 4r^{-1}(\alpha_1 - \alpha_4) + r^{-2}(\alpha_{22} + \alpha_2\cot\theta + 2\alpha) = 2(r^{-1}\delta_1 + r^{-2}\delta) - U + 2T. \qquad \text{(A.23)}$$

From equation (A.22) we have

$$(\beta \sin\theta)_2 = \sin\theta\{r(-\alpha_1+\alpha_4+\delta_1)+2(-\alpha+\delta)+r^2T\}. \tag{A.24}$$

Thus, knowing δ from the wave equation (A.21) enables α to be evaluated directly from equation (A.23); this, in turn, allows β to be readily calculated directly from equation (A.24). Accordingly equations (A.21), (A.23) and (A.24) may be regarded as constituting the formal solution of the (ps) M equations (A.19) to (A.22).

It should be remarked that any solution of equations (A.19) to (A.22), found with the aid of equations (A.21), (A.23) and (A.24), must satisfy all the four equations (A.19) to (A.22), the Galilean condition $\kappa_i = 0$ at spatial infinity and the regularity condition on the rotation axis Oz, the latter being met if

$$\alpha,\ \beta \operatorname{cosec}\theta,\ \delta \text{ are of class } C^2 \text{ near } \sin\theta = 0. \tag{A.25}$$

This must be arranged by assigning appropriate values to the arbitrary functions of integration inherent in the solution (A.21), (A.23) and (A.24).

Appendix 4. The advanced multipole wave solution corresponding to the Bondi metric

In Galilean coordinates, the advanced multipole wave solution of equations (3.18) for the source of § 2, obtained by putting $\alpha = 0$, $\beta = 1$ and (3.21) in (3.17) is

$$\begin{aligned} \kappa_1 &= \kappa_2 = 0 \qquad \kappa_3 = \epsilon\{-ar^{-1}h'+\mathrm{O}(a^2)\} \\ \kappa_4 &= \epsilon\{r^{-1}+a\cos\theta\,(-r^{-1}h'+r^{-2}h)+\mathrm{O}(a^2)\} \end{aligned} \tag{A.26}$$

where $h \equiv h(t+r)$ and a prime means differentiation with respect to $t+r$†. The coordinate transformation (5.4) brings (A2.6) into

$$\begin{aligned} \kappa_1 &= \epsilon[r^{-1}+a\cos\theta\{-2r^{-1}h'(u+2r)+r^{-2}h(u+2r)\}+\mathrm{O}(a^2)] \\ \kappa_2 &= \epsilon\{a\sin\theta\, h'(u+2r)+\mathrm{O}(a^2)\} \qquad \kappa_3 = 0 \\ \kappa_4 &= \epsilon[r^{-1}+a\cos\theta\{-r^{-1}h'(u+2r)+r^{-2}h(u+2r)\}+\mathrm{O}(a^2)] \end{aligned} \tag{A.27}$$

which is the advanced multipole wave solution (for the source of § 2) that corresponds to the Bondi metric (5.1)‡. The advanced dipole wave contribution in (A.27) is, in the notation (5.7)

$$\overset{(11)}{\alpha} = -\cos\theta\,(r^{-1}h)_1 \qquad \overset{(11)}{\beta} = \tfrac{1}{2}\sin\theta\, r^{-1}h_1 \qquad \overset{(11)}{\delta} = -\tfrac{1}{2}\cos\theta\, r(r^{-2}h)_1 \tag{A.28}$$

in which $h \equiv h(u+2r)$. From the second part of (3.12) and (A.27), the incoming dipole wave contribution to the nonzero components of the field tensor are readily found to be

$$\begin{aligned} \overset{(11)}{H}_{12} &= -\tfrac{1}{2}\sin\theta\, r(r^{-1}h)_{11} \qquad \overset{(11)}{H}_{14} = -\cos\theta\,(r^{-2}h)_1 \\ \overset{(11)}{H}_{24} &= \tfrac{1}{4}\sin\theta\,(h_{11}-2r^{-1}h_1+4r^{-2}h) \end{aligned} \tag{A.29}$$

with $h \equiv h(u+2r)$.

† A remark similar to that made in the footnote on page 621 concerning equations (3.22) applies to equations (A.26).

‡ A comment analogous to that given in the footnote on page 624 in connection with equations (5.5) is relevant to equations (A.27).

References

BONDI, H., 1960, *Nature*, **186,** 535.

BONNOR, W. B., 1959, *Phil. Trans. R. Soc.* A, **251,** 233–71.

BONNOR, W. B., and ROTENBERG, M. A., 1966, *Proc. R. Soc.* A, **289,** 247–74.

COUCH, W. E., TORRENCE, R. J., JANIS, A. I., and NEWMAN, E. T., 1968, *J. math. Phys.*, **9,** 484–96.

EDDINGTON, A. S., 1924, *The Mathematical Theory of Relativity* (London: Cambridge University Press).

HUNTER, A. J., and ROTENBERG, M. A., 1969, *J. Phys.* A: *Gen. Phys.*, **2,** 34–49.

KÜNZLE, H. P., 1968, *Proc. Camb. Phil. Soc.*, **64,** 779–85.

ROTENBERG, M. A., 1964, *PhD Thesis*, University of London.

—— 1966, *Proc. R. Soc.* A, **293,** 408–22.

—— 1967, *Commun. math. Phys.*, **5,** 23–41.

—— 1968, *J. Phys.* A: *Gen. Phys.*, **1,** 280–94.